STANLEY DECOMPOSITIONS AND LOCALIZATION

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Abstract. We study the behavior of Stanley depth under the operation of localization with respect to a variable.

Introduction

Let K be a field, $S = K[x_1, \dots, x_n]$ be the polynomial ring in n variables over K and $I \subset S$ a monomial ideal. Stanley depth of S/I is denoted by $\operatorname{sdepth} S/I$, see Section 2 for its definition. The Stanley depth is an important combinatorial invariant of S/I studied in [5], [6], [7], [8]. The interest in this subject arises in part from the so-called Stanley conjecture which asserts that $\operatorname{sdepth} S/I \geq \operatorname{depth} S/I$.

The purpose of this note is to study the behavior of $\operatorname{sdepth} S/I$ under the operation of localization with respect to a variable. The effect of localization of a monomial ideal with respect to a variable, say x_n, is, up to a flat extension, the same as applying the K-algebra homomorphism $\varphi : S \to T = K[x_1, \dots, x_{n-1}]$ given by $x_n \mapsto 1$. This is explained in Section 1.

Many, but not all, Stanley decompositions arise as prime filtrations. In Section 2 we show how prime filtrations behave under localization, see Proposition 2.1. As a consequence we show in Corollary 2.2 that pretty clean filtrations induce under localization again pretty clean filtrations. This implies in particular that if Stanley's conjecture holds for S/I, then it holds for the localization as well. As an immediate consequence of Proposition 2.1 we show that $\operatorname{fdepth} T/\varphi(I) \geq \operatorname{fdepth}(S/I) - 1$, where fdepth, introduced in [6], is an invariant of S/I related to prime filtrations. This invariant is of interest since one always has $\operatorname{fdepth} S/I \leq \operatorname{sdepth} S/I, \operatorname{depth} S/I$.

The main purpose of Section 3 is to prove an inequality analogue to that for the fdepth. In fact, we show in Corollary 3.2 that $\operatorname{sdepth} T/\varphi(I) \geq \operatorname{sdepth}(S/I) - 1$. Easy examples show that the inequality is often strict. On the other hand, we also give an example for which $\operatorname{sdepth} T/\varphi(I) > \operatorname{sdepth}(S/I)$.

When $I = I_\Delta$ is the Stanley-Reisner ideal of a simplicial complex Δ we get in particular that $\operatorname{sdepth} K[\operatorname{link}_\Delta(\{n\})] \geq \operatorname{sdepth} K[\Delta] - 1$, where $K[\Delta] = S/I$ (see Lemma 3.7).

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1. Localization of monomial ideals

Let K be a field and $S = K[x_1, \dots, x_n]$ be the polynomial ring in n variables over K, and let $I \subset S$ a be a monomial ideal. Suppose that I is generated by the monomials $u_1, \dots, u_m$ with $u_i = \prod_{j=1}^{n} x_j^{a_{ij}}$. We denote, as usual, by S_{x_n} the localization of S with respect to the element x_n. Notice that S_{x_n} has a K-basis consisting of all monomials of the form

$$x_1^{a_1} x_2^{a_2} \cdots x_{n-1}^{a_{n-1}} x_n^{a_n} \quad \text{with} \quad a_i \in \mathbb{Z}_{\geq 0} \quad \text{and} \quad a_n \in \mathbb{Z}.$$

In other words,

$$S_{x_n} = K[x_n, x_n^{-1}][x_1, \dots, x_{n-1}] = K[x_n, x_n^{-1}] \otimes_K T,$$

where $T = K[x_1, \dots, x_{n-1}]$.

The extension ideal IS_{x_n} is the ideal in S_{x_n} which is generated by the monomials $u_i' = \prod_{j=1}^{n-1} x_j^{a_{ij}}$, because the last variable becomes a unit.

Let $\varphi : S \to T$ be the K-algebra homomorphism with $x_i \mapsto x_i$ for $i = 1, \dots, n-1$ and $x_n \mapsto 1$, then $\varphi(u_i) = u_i'$ for all i and we see that IS_{x_n} is the extension ideal of $\varphi(I)$ under the flat extension $T \to K[x_n, x_n^{-1}] \otimes_K T = S_{x_n}$.

2. Localization of prime filtrations

Let K be a field and $S = K[x_1, \dots, x_n]$ be the polynomial ring in n variables over K. Let $I \subset S$ be a monomial ideal. A *prime filtration* of S/I is a chain of monomial ideals

$$\mathcal{P} : I = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_r = S$$

such that there are isomorphisms of $\mathbb{Z}^n$-graded S-modules

$$I_j/I_{j-1} \cong (S/P_j)(-a_j) \quad \text{for} \quad j = 1, 2, \dots, r,$$

where P_j is a monomial prime ideal and $a_j \in \mathbb{Z}^n$. The set $\{P_1, \dots, P_r\}$ is called the *support* of $\mathcal{P}$ and denoted $\mathrm{Supp}(\mathcal{P})$.

We consider the K-algebra homomorphism $\varphi \colon S \to T = K[x_1, \dots, x_{n-1}]$, introduced in the previous section, with $x_i \mapsto x_i$ for $i = 1, \dots, n-1$ and $x_n \mapsto 1$. We will also consider the projection map $\pi \colon \mathbb{Z}^n \to \mathbb{Z}^{n-1}$ which assigns to each $a = (a_1, \dots, a_n)$ in $\mathbb{Z}^n$ the vector $a' = \pi(a) = (a_1, \dots, a_{n-1})$.

Proposition 2.1. *Let $I \subset S$ be a monomial ideal, and let $\mathcal{P}$ be a prime filtration of S/I as above. We set $J = \varphi(I)$ and $J_j = \varphi(I_j)$ for all I_j in the prime filtration. Then we get the filtration*

$$J = J_0 \subseteq J_1 \subseteq J_2 \subseteq \cdots \subseteq J_r = T$$

with

$$J_j/J_{j-1} \cong \begin{cases} (T/P_j')(-a_j'), & \text{if } x_n \notin P_j, \\ 0, & \text{if } x_n \in P_j, \end{cases}$$

where $P_j' \subset T$ is the monomial prime ideal in T such that $P_j = P_j'S$.

Proof. The statement of the proposition follows once we can show the following: Let $I \subset J$ be monomial ideals in S such that $J/I \cong (S/P)(-a)$ where P is a monomial prime ideal and $a \in \mathbb{Z}^n_{\geq 0}$. Then

$$\varphi(J)/\varphi(I) \cong \begin{cases} (T/P')(-a'), & \text{if } x_n \notin P, \\ 0, & \text{if } x_n \in P, \end{cases}$$

We have $J/I \cong (S/P)(-a)$ if and only if $J = (I, x^a)$ and $I :_S x^a = P$. Since

$$\varphi(J) = \varphi(I, x^a) = (\varphi(I), x^{a'}),$$

we see that

$$\varphi(J)/\varphi(I) \cong (\varphi(I), x^{a'})/\varphi(I)) \cong (T/(\varphi(I) :_T x^{a'}))(-a'). \tag{1}$$

Next we claim that $\varphi(I :_S x^a) = (\varphi(I) :_T x^{a'})$. Suppose this is true, then we get

$$(\varphi(I) :_T x^{a'}) = \varphi(P) = \begin{cases} P', & \text{if } x_n \notin P, \\ T, & \text{if } x_n \in P, \end{cases}$$

Hence the desired result follows.

It remains to prove the claim: let $I = (u_1, \ldots, u_m)$ with $u_i = x^{a_i} = \prod_{j=1}^n x_j^{a_{ij}}$. Then

$$\begin{aligned} I :_S x^a &= (x^{a_1}/\gcd(x^{a_1}, x^a), \ldots, x^{a_m}/\gcd(x^{a_m}, x^a)) \\ &= (\prod_{j=1}^n x_j^{a_{1j}-\min\{a_{1j}, a_j\}}, \cdots, \prod_{j=1}^n x_j^{a_{mj}-\min\{a_{mj}, a_j\}}). \end{aligned}$$

It follows that

$$\begin{aligned} \varphi(I :_S x^a) &= (\varphi(\prod_{j=1}^n x_j^{a_{1j}-\min\{a_{1j}, a_j\}}), \cdots, \varphi(\prod_{j=1}^n x_j^{a_{mj}-\min\{a_{mj}, a_j\}})) \\ &= (\prod_{j=1}^{n-1} x_j^{a_{1j}-\min\{a_{1j}, a_j\}}, \cdots, \prod_{j=1}^{n-1} x_j^{a_{mj}-\min\{a_{mj}, a_j\}}) \\ &= (x^{a'_1}/\gcd(x^{a'_1}, x^{a'}), \ldots, x^{a'_m}/\gcd(x^{a'_m}, x^{a'})) \\ &= (\varphi(x^{a_1})/\gcd(\varphi(x^{a_1}), \varphi(x^a)), \ldots, \varphi(x^a)/\gcd(\varphi(x^{a_m}), \varphi(x^a))) \\ &= \varphi(I) :_T x^{a'}. \end{aligned}$$

□

Let K be a field and $S = K[x_1, \ldots, x_n]$ be a polynomial ring. Let $I \subset S$ be a monomial ideal. A prime filtration

$$\mathcal{P} : I = I_0 \subset I_1 \subset \cdots \subset I_r = S$$

of S/I such that $I_j/I_{j-1} \cong (S/P_j)(-a_j)$ is said to be *clean* (see [3]) if $\operatorname{Supp}(\mathcal{P}) = \operatorname{Min}(S/I)$, where $\operatorname{Min}(S/I)$ denotes the set of minimal prime ideals of I. Equivalently, $(\mathcal{P})$ is clean, if there is no containment between the elements in $\operatorname{Supp}(\mathcal{P})$, see [4]. A monomial ideal I is said to be *clean* if S/I has a clean filtration. The prime filtration $\mathcal{P}$ is said to be *pretty clean* if for all $i < j$ the inclusion $P_i \subset P_j$ implies

$P_i = P_j$ (see [4]). A monomial ideal I is said to be *pretty clean* if S/I has a pretty clean filtration.

Let $I \subset S$ be a monomial ideal. We denote by $I^c \subset S$ the K linear subspace of S generated by all monomials which do not belong to I. Then $S = I \oplus I^c$ and $S/I \cong I^c$ as K-linear spaces. If $u \in S$ is a monomial and $Z \subset \{x_1, \ldots, x_n\}$, the K-subspace $uK[Z]$ whose basis consists of all monomials uv with $v \in K[Z]$ is called a *Stanley space* of dimension $|Z|$. A decomposition $\mathcal{D}$ of I^c as a finite direct sum of *Stanley spaces* is called a *Stanley decomposition* of S/I. The minimal dimension of a Stanley spaces in $\mathcal{D}$ is called the *Stanley depth* of $\mathcal{D}$ and is denoted by $\operatorname{sdepth} \mathcal{D}$. Finally we define $\operatorname{sdepth} S/I$ by

$$\operatorname{sdepth} S/I = \max\{\operatorname{sdepth} \mathcal{D} : \ \mathcal{D} \quad \text{is a Stanley decomposition of } S/I\}$$

In [9] Stanley conjectures that for any monomial ideal $I \subset S$ one has $\operatorname{sdepth} S/I \geq \operatorname{depth} S/I$. The monomial ideal I is said to be a *Stanley ideal* if Stanley's conjecture holds for S/I. It is shown in [4] that a pretty clean ideal is a Stanley ideal.

As a consequence of the previous result we have

Corollary 2.2. *Let $I \subset S$ be a monomial ideal. If I is (pretty) clean, then $\varphi(I) \subset T$ is (pretty) clean. In particular, if I is pretty clean, then $\varphi(I) \subset T$ is a Stanley ideal.*

Proof. We refer to the the hypotheses and notation of Proposition 2.1, and assume in addition that the filtration $\mathcal{P}$ of S/I is (pretty) clean. The filtration of J given in Proposition 2.1 can be modified to give a prime filtration of T/J (by omitting for all $i > 0$ those J_i for which $J_{i-1} = J_i$) whose support is a subset of $\operatorname{Supp}(\mathcal{P})$. From this, all assertions follow immediately. $\square$

Let $\mathcal{F}\colon I = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_r = S$ be a prime filtration with $I_j/I_{j-1} \cong S/P_j(-a_j)$. Then

$$\mathcal{D}(F) : S/I = \bigoplus_{j=1}^{r} u_i K[Z_i]$$

is a Stanley decomposition of S/I, where $u_i = x^{a_i}$ and $Z_i = \{x_j : \ x_j \notin P_i\}$ (see [4]). Thus if we set $\operatorname{fdepth} \mathcal{F} = \min\{\dim S/P_1, \ldots, \dim S/P_r\}$ and

$$\operatorname{fdepth} S/I = \max\{\operatorname{fdepth} \mathcal{F}\colon \mathcal{F} \text{ is a prime filtration of } S/I\},$$

then see that $\operatorname{fdepth} S/I \leq \operatorname{sdepth} S/I$.

As an immediate consequence of Proposition 2.1 we obtain

Corollary 2.3. *Let $I \subset S$ be a pretty clean monomial ideal. Then*

$$\operatorname{fdepth} T/\varphi(I) \geq \operatorname{fdepth} S/I - 1.$$

3. Localizations and Stanley decompositions

The purpose of this section is to prove an inequality for the sdepth similar to that for the fdepth given in Corollary 2.3 in Section 2. The desired inequality will be a consequence of

Theorem 3.1. *Let $\mathcal{D} : S/I = \bigoplus_{i=1}^{r} u_i K[Z_i]$ be a Stanley decomposition of S/I then $\mathcal{D}' : T/\varphi(I) = \bigoplus\limits_{x_n \in Z_i} \varphi(u_i)K[Z_i \setminus \{x_n\}]$ is a Stanley decomposition of $T/\varphi(I)$.*

Proof. Firstly we prove that

$$\varphi(u_i)K[Z_i \setminus \{x_n\}] \cap \varphi(u_j)K[Z_j \setminus \{x_n\}] = \{0\}$$

for $i \neq j$ and $x_n \in Z_i, Z_j$. Suppose on the contrary that there exists a monomial $u \in T$ such that

$$u \in \varphi(u_i)K[Z_i \setminus \{x_n\}] \cap \varphi(u_j)K[Z_j \setminus \{x_n\}],$$

that is

$$u = \varphi(u_i)f_i = \varphi(u_j)f_j,$$

for some monomials $f_i \in K[Z_i \setminus \{x_n\}], f_j \in K[Z_j \setminus \{x_n\}]$. It follows that $ux_n^a \in u_iK[Z_i]$ and $ux_n^a \in u_jK[Z_j]$ for some $a \in \mathbb{N}$ sufficiently large. Hence

$$ux_n^a \in u_iK[Z_i] \cap u_jK[Z_j],$$

that is a contradiction.

Let $u \in T \setminus \varphi(I)$ be a monomial. We claim that there exists $i \in [r]$ such that $u \in \varphi(u_i)K[Z_i \setminus \{x_n\}]$. Note that $\varphi(u) = u$ and $u \in I^c$ because otherwise $u \in \varphi(I)$, which is a contradiction. This implies that there exist $i \in [r]$ such that $u \in u_iK[Z_i]$. Hence

$$\varphi(u) = u \in \varphi(u_i)K[Z_i \setminus \{x_n\}].$$

Remains to show that we may choose i such that $x_n \in Z_i$. If $x_n \notin Z_i$ then there exists $j \in [r]$ such that $i \neq j$ and $t > s = \deg_{x_n} u_i$ such that $ux_n^t \in u_jK[Z_j]$ with $x_n \in Z_j$. Indeed, we have $ux_n^t = u_j g$, where $g \in K[Z_j]$ is a monomial. It follows that x_n^t does not divide u_j because $t > s$, so x_n divides g. This implies $x_n \in Z_j$. □

Corollary 3.2.

$$\operatorname{sdepth} T/\varphi(I) \geq \operatorname{sdepth} S/I - 1.$$

Proof. In the above theorem, let $\mathcal{D}$ be a Stanley decomposition of S/I such that $\operatorname{sdepth} \mathcal{D} = \operatorname{sdepth} S/I$. Then we have

$$\operatorname{sdepth} T/\varphi(I) \geq \operatorname{sdepth} \mathcal{D}' = \operatorname{sdepth} S/I - 1.$$

□

Example 3.3. Let $I = (xy) \subset S = K[x, y]$ be an ideal, $\mathcal{D} : S/I = xK[x] \oplus K[y]$ is a Stanley decomposition of S/I. Thus $\operatorname{sdepth} \mathcal{D} = 1$. After applying the map φ defined by $x \to 1$, $\mathcal{D}' : T/\varphi(I) = K$ is a Stanley decomposition of $T/\varphi(I)$ and $\operatorname{sdepth} \mathcal{D}' = 0$.

Example 3.4. Let $I = (x^2, xy)$ be an ideal of $S = K[x, y]$. A Stanley decomposition of S/I is $\mathcal{D} : S/I = xK \oplus K[y]$. Thus for φ given by $y \to 1$, $\mathcal{D}' : T/\varphi(I) = K$ is a Stanley decomposition of $T/\varphi(I)$. Here $\operatorname{sdepth} S/I = 0$ and $\operatorname{sdepth} T/\varphi(I) = 0$.

Example 3.5. Let $I = (xyz) \subset S = K[x, y, z]$ be an ideal. Then $\mathcal{D}$: $S/I = K[x, z] \oplus yK[x, y] \oplus zyK[y, z]$ is a Stanley decomposition of S/I with $\operatorname{sdepth} \mathcal{D} = 2$. After applying the map φ given by $z \to 1$, $\mathcal{D}'$: $T/\varphi(I) = K[x] \oplus yK[y]$ is a Stanley decomposition of $T/\varphi(I)$ and $\operatorname{sdepth} \mathcal{D}' = 1$.

The following example shows that the inequality in Corollary 3.2 may be strict.

Example 3.6. Let $I = (xy, xz, xw) \subset S = K[x, y, z, w]$ be the squarefree monomial ideal. Then

$$S/I = xK[x] \oplus K[y, z] \oplus wK[y, z, w]$$

is a Stanley decomposition of S/I. Thus $\operatorname{sdepth} S/I \geq 1$. By using partitions of the characteristic poset of S/I (see [5]), one can show that indeed $\operatorname{sdepth} S/I = 1$. After applying φ we get $\varphi(I) = (x) \subset K[x, y, z]$ and $T/\varphi(I) = K[x, y, z]/(x) \cong K[y, z]$. Hence $\operatorname{sdepth} T/\varphi(I) = 2$. So we get

$$\operatorname{sdepth} T/\varphi(I) > \operatorname{sdepth} S/I.$$

We conclude this section by interpreting the inequality in Corollary 3.2 for squarefree monomial ideals in terms of simplicial complexes.

Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over the field K and $I \subset S$ an ideal generated by squarefree monomials. Let $\triangle$ be a simplicial complex on he vertex set $[n]$ such that I is the Stanley-Reisner ideal I_Δ associated to Δ and $K[\Delta] = S/I$. As above consider $T/\varphi(I)$.

Lemma 3.7. $T/\varphi(I) = K[\operatorname{link}_\Delta(\{n\})]$.

Proof. It is enough to show that $\varphi(I_\Delta) = I_{\operatorname{link}_\Delta(\{n\})}$. Let $G \subset [n-1]$ be such that $x^G \in I_{\operatorname{link}_\Delta(\{n\})}$. This implies that $G \notin \operatorname{link}_\Delta(\{n\})$ and so $G \cup \{n\} \notin \Delta$. Hence $x^{G\cup\{n\}} \in I_\Delta$. This implies that $x^G \in \varphi(I_\Delta)$.

A square free monomial of I_Δ has the form x^H with $H \subset [n]$ and $H \notin \Delta$. If $n \notin H$ then $x^H = \varphi(x^H) \in \varphi(I_\Delta)$. Since $H \notin \Delta$, we get that $H \cup \{n\} \notin \Delta$. Then $H \notin \operatorname{link}_\Delta(\{n\})$ and so $x^H \in I_{\operatorname{link}_\Delta(\{n\})}$. If $n \in H$ then $x^{H\setminus\{n\}} = \varphi(x^H) \in \varphi(I_\Delta)$. As $(H \setminus \{n\}) \cup \{n\} = H \notin \Delta$ we get $H \setminus \{n\} \notin \operatorname{link}_\Delta(\{n\})$. Thus $x^{H\setminus\{n\}} \in I_{\operatorname{link}_\Delta(\{n\})}$. □

Corollary 3.8.

$$\operatorname{sdepth} K[\operatorname{link}_\Delta(\{n\})] \geq \operatorname{sdepth} K[\Delta] - 1.$$

Proof. The result follows from the above lemma and Corollary 3.2. □

Corollary 3.9. *For any subset $F \subset [n]$,*

$$\operatorname{sdepth} K[\operatorname{link}_\Delta(F)] \geq \operatorname{sdepth} K[\Delta] - |F|.$$

Proof. We may assume that $n \in F$. Apply induction on $|F|$, the case $|F| = 1$ was done in the previous corollary. Suppose $|F| > 1$. Then by the same corollary we get $\operatorname{sdepth}(K[\operatorname{link}_\Delta(\{n\})]) \geq \operatorname{sdepth}(K[\Delta]) - 1$. Apply induction hypothesis for $\operatorname{link}_\Delta(\{n\})$ and $F' = F \setminus \{n\}$. Then

$$\begin{aligned}
\operatorname{sdepth} K[\operatorname{link}_\Delta(F)] &= \operatorname{sdepth} K[\operatorname{link}_{\operatorname{link}_\Delta(\{n\})}(F')] \\
&\geq \operatorname{sdepth} K[\operatorname{link}_\Delta(\{n\})] - |F'| \\
&\geq (\operatorname{sdepth} K[\Delta] - 1) - |F'| \\
&= \operatorname{sdepth} K[\Delta] - |F|.
\end{aligned}$$

□

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